

# ON THE TERNARY GOLDBACH PROBLEM WITH PRIMES IN INDEPENDENT ARITHMETIC PROGRESSIONS

Karin Halupczok

## Abstract

We show that for every fixed  $A > 0$  and  $\theta > 0$  there is a  $\vartheta = \vartheta(A, \theta) > 0$  with the following property. Let  $n$  be odd and sufficiently large, and let  $Q_1 = Q_2 := n^{1/2}(\log n)^{-\vartheta}$  and  $Q_3 := (\log n)^\theta$ . Then for all  $q_3 \leq Q_3$ , all reduced residues  $a_3 \bmod q_3$ , almost all  $q_2 \leq Q_2$ , all admissible residues  $a_2 \bmod q_2$ , almost all  $q_1 \leq Q_1$  and all admissible residues  $a_1 \bmod q_1$ , there exists a representation  $n = p_1 + p_2 + p_3$  with primes  $p_i \equiv a_i \pmod{q_i}$ ,  $i = 1, 2, 3$ .

## 1 Introduction and results

### 1.1 Preliminaries

Let  $n$  be a sufficiently large integer, and for every  $i = 1, 2, 3$  let  $a_i, q_i$  be relatively prime integers with  $q_i \geq 1$  and  $0 \leq a_i < q_i$ .

We consider the ternary Goldbach problem of writing  $n$  as

$$n = p_1 + p_2 + p_3$$

with primes  $p_1, p_2$  and  $p_3$  satisfying the three congruences

$$p_i \equiv a_i \pmod{q_i}, \quad i = 1, 2, 3.$$

A necessary condition for solvability is

$$n \equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)},$$

where  $(q_1, q_2, q_3)$  denotes the greatest common divisor of the  $q_i$ . Otherwise no such representation of  $n$  is possible.

We precise our consideration in the following way. Let

$$J_3(n) := \sum_{\substack{m_1+m_2+m_3=n \\ m_i \equiv a_i \pmod{q_i}, \\ i=1,2,3}} \Lambda(m_1) \Lambda(m_2) \Lambda(m_3),$$

where  $\Lambda$  is von Mangoldt's function.  $J_3(n)$  goes closely with the number of representations of  $n$  in the way mentioned.

In this paper we prove that the deviation of  $J_3(n)$  from its expected main term is uniformly small for large moduli, namely:

**Theorem 1.** *For every fixed  $A > 0$  and  $\theta > 0$  there is a  $\vartheta = \vartheta(A, \theta) > 0$  such that for all  $q_3 \leq (\log n)^\theta$  and  $a_3$  with  $(a_3, q_3) = 1$  we have*

$$\sum_{q_2 \leq \frac{n^{1/2}}{(\log n)^\vartheta}} \max_{\substack{a_2 \\ (a_2, q_2)=1}} \sum_{q_1 \leq \frac{n^{1/2}}{(\log n)^\vartheta}} \max_{\substack{a_1 \\ (a_1, q_1)=1}} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \ll \frac{n^2}{(\log n)^A}.$$

*The  $O$ -constant depends on the parameters  $A$  and  $\theta$ .*

Here  $\mathcal{S}_3(n)$  denotes the singular series for this special Goldbach problem and depends on  $a_i$  and  $q_i$  likewise  $J_3(n)$  does.

We set  $\mathcal{S}_3(n) = 0$  if  $n \not\equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)}$ , the case where trivially  $J_3(n) = 0$  occurs. Then a summand  $= 0$  in the formula of Theorem 1 is given, therefore we can assume in the proof without loss of generality that  $n \equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)}$  holds. We refer to this as "general condition", under this,  $\mathcal{S}_3(n)$  is defined and investigated later in paragraphs 2.2 and 2.3.

**Definition.** For any given  $q_1, q_2, q_3$  we call a triplet  $a_1, a_2, a_3$  of residues mod  $q_1, q_2, q_3$  *admissible* for  $q_1, q_2, q_3$ , if  $(a_i, q_i) = 1$  for  $i = 1, 2, 3$ , if  $n \equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)}$  and if  $\mathcal{S}_3(n) > 0$ .

For given  $q_3, a_3, q_2, a_2$  and  $q_1$  we call  $a_1$  *admissible*, if  $a_1, a_2, a_3$  is admissible for  $q_1, q_2, q_3$ . For given  $q_3, a_3, q_2$  we call  $a_2$  *admissible*, if there exists an admissible  $a_1$  for every positive integer  $q_1$ .

We prove in paragraph 2.3

**Lemma 1.** *If  $n$  is odd, then for given  $q_3, a_3$  with  $(a_3, q_3) = 1$  and  $q_2$  there exists an admissible  $a_2$  (such that for every  $q_1$  there exists an admissible  $a_1$ ). For even  $n$  and given  $q_1, q_2, q_3$  there exists no admissible triplet  $a_1, a_2, a_3$ .*

Theorem 1 provides

**Theorem 2.** *Let  $A, \theta, \vartheta > 0$  as above and  $n \in \mathbb{N}$  odd and sufficiently large. Let  $Q_1, Q_2 := n^{1/2}(\log n)^{-\vartheta}$ ,  $Q_3 := (\log n)^\theta$ . Then for all  $q_3 \leq Q_3$ , all  $a_3$ , almost all  $q_2 \leq Q_2$ , all admissible  $a_2$ , almost all  $q_1 \leq Q_1$  and all admissible  $a_1$  there exists a representation  $n = p_1 + p_2 + p_3$  with primes  $p_i \equiv a_i \pmod{q_i}$ ,  $i = 1, 2, 3$ . Here the number of exceptions for  $q_2$  is  $\ll Q_2(\log n)^{-A}$  resp. for  $q_1$  is  $\ll Q_1(\log n)^{-A}$ .*

Theorem 2 as corollary of Theorem 1 is proved in section 6.

Theorem 1 is shown by the circle method. It seems that it also should hold with the larger bound  $q_3 \leq n^{1/2}(\log n)^{-\vartheta}$ , which is the case on the major arcs. It is not possible to achieve this on the minor arcs by the given methods.

**Notation.** We denote by  $\varphi$ ,  $\mu$ ,  $\Lambda$  and  $\tau$  the functions of Euler, Möbius, von Mangoldt and the divisor function. Other occurring functions are given in their context. By  $q_i \sim Q_i$  we abbreviate  $Q_i < q_i \leq 2Q_i$ . By  $p$  and  $p_i$  we denote primes. As usual,  $e(\alpha) := e^{2\pi i \alpha}$  for  $\alpha \in \mathbb{R}$ .

## 1.2 Proceeding by the circle method

Let  $A > 0$  and  $\theta > 0$ . Let  $R := (\log n)^B$  with  $B = B(A, \theta) := \max\{A + \eta + 3, D(8A + 2\theta + 74)\}$ , where  $\eta > 0$  is some absolute constant (see end of paragraph 2.2), and  $D(8A + 2\theta + 74) > 0$  is some constant depending just on  $A$  and  $\theta$ , its definition is given in the proof of Lemma 5. Further let  $\vartheta > \max\{A + 4B + 16, \theta + A + 3\}$ , so  $\vartheta$  depends also on  $A$  and  $\theta$ .

We define major arcs  $\mathfrak{M} \subseteq \mathbb{R}$  by

$$\mathfrak{M} := \bigcup_{q \leq R} \bigcup_{\substack{0 < a < q \\ (a, q) = 1}} \left] \frac{a}{q} - \frac{R}{qn}, \frac{a}{q} + \frac{R}{qn} \right[$$

and minor arcs by  $\mathfrak{m} := \left] -\frac{R}{n}, 1 - \frac{R}{n} \right[ \setminus \mathfrak{M}$ .

For  $\alpha \in \mathbb{R}$  and  $j = 1, 2, 3$  let

$$S_j(\alpha) := \sum_{\substack{m \leq n \\ m \equiv a_j \pmod{q_j}}} \Lambda(m) e(\alpha m).$$

From the orthogonal relations for  $e(\alpha m)$  it follows that

$$J_3(n) = \int_{-\frac{R}{n}}^{1-\frac{R}{n}} S_1(\alpha) S_2(\alpha) S_3(\alpha) e(-n\alpha) d\alpha.$$

Analogously, denote for  $m \leq n$

$$J_2(m) := \sum_{\substack{m_2+m_3=m \\ m_2 \equiv a_2 \pmod{q_2} \\ m_3 \equiv a_3 \pmod{q_3}}} \Lambda(m_2) \Lambda(m_3) = \int_{-\frac{R}{n}}^{1-\frac{R}{n}} S_2(\alpha) S_3(\alpha) e(-m\alpha) d\alpha.$$

By

$$J_3^{\mathfrak{M}}(n) := \int_{\mathfrak{M}} S_1(\alpha) S_2(\alpha) S_3(\alpha) e(-n\alpha) d\alpha$$

and

$$J_2^{\mathfrak{M}}(m) := \int_{\mathfrak{M}} S_2(\alpha) S_3(\alpha) e(-m\alpha) d\alpha$$

denote the values of  $J_3(n)$  and  $J_2(m)$  on the major arcs  $\mathfrak{M}$  and by

$$J_3^{\mathfrak{m}}(n) := J_3(n) - J_3^{\mathfrak{M}}(n), \quad J_2^{\mathfrak{m}}(m) := J_2(m) - J_2^{\mathfrak{M}}(m)$$

the values on the minor arcs  $\mathfrak{m}$ .

Concerning the major arcs we get

**Theorem 3.** *For  $Q_1, Q_2, Q_3 \leq n^{1/2}/(\log n)^\vartheta$  we have*

$$\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{M}} := \sum_{\substack{q_i \sim Q_i, \\ i=1,2,3}} \max_{\substack{a_i, (a_i, q_i)=1, \\ i=1,2,3}} \left| J_3^{\mathfrak{M}}(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \ll \frac{n^2}{(\log n)^{A+3}}.$$

We prove Theorem 3 in the following section 2.

In section 3 a lemma containing a special form of Montgomery's sieve is proven. Section 4 delivers a proof of Theorem 1 using Theorem 3 and the lemma from section 3. Further used lemmas concerning estimations on the minor arcs are proven afterwards in section 5.

## 2 Estimations on the major arcs

### 2.1 Getting the main term and the error term

We have

$$J_3^{\mathfrak{M}}(n) = \sum_{q \leq R} \sum_{\substack{0 < a < q \\ (a, q) = 1}} I(a, q),$$

where

$$I(a, q) := \int_{-\frac{R}{qn}}^{\frac{R}{qn}} S_1\left(\frac{a}{q} + \alpha\right) S_2\left(\frac{a}{q} + \alpha\right) S_3\left(\frac{a}{q} + \alpha\right) e\left(-n\left(\frac{a}{q} + \alpha\right)\right) d\alpha.$$

For  $j = 1, 2, 3$  we have for  $\alpha \in [-\frac{R}{qn}, \frac{R}{qn}]$

$$\begin{aligned} S_j\left(\frac{a}{q} + \alpha\right) &= \sum_{\substack{m \leq n \\ m \equiv a_j(q_j)}} \Lambda(m) e(\alpha m) e\left(\frac{a}{q}m\right) \\ &= \sum_{\substack{m \leq n \\ m \equiv a_j(q_j) \\ (m, q) = 1}} \Lambda(m) e(\alpha m) e\left(\frac{a}{q}m\right) + \sum_{\substack{m \leq n \\ m \equiv a_j(q_j) \\ (m, q) > 1}} \Lambda(m) e(\alpha m) e\left(\frac{a}{q}m\right) \\ &= \sum_{\substack{1 \leq k \leq q \\ (k, q) = 1}} \sum_{\substack{m \leq n \\ m \equiv a_j(q_j) \\ m \equiv k(q)}} \Lambda(m) e(\alpha m) e\left(\frac{a}{q}k\right) + O((\log n)^2) \end{aligned}$$

since

$$\sum_{\substack{m \leq n \\ m \equiv a_j(q_j) \\ (m, q) > 1}} \Lambda(m) = \sum_{\substack{p^e \leq n \\ p^e \equiv a_j(q_j) \\ p|q}} \log p \leq \sum_{p|q} \log p \cdot \frac{\log n}{\log p} \ll \log n \sum_{p|q} 1 \ll (\log n)^2.$$

So

$$S_j\left(\frac{a}{q} + \alpha\right) = \sum_{\substack{1 \leq k \leq q \\ (k, q) = 1 \\ k \equiv a_j((q_j, q))}} e\left(\frac{a}{q}k\right) T_{j, k}(\alpha) + O((\log n)^2)$$

with

$$T_{j,k}(\alpha) := \sum_{\substack{m \leq n \\ m \equiv a_j(q_j) \\ m \equiv k(q)}} \Lambda(m) e(\alpha m) = \sum_{\substack{m \leq n \\ m \equiv f_{j,k}([q_j, q])}} \Lambda(m) e(\alpha m).$$

Here  $T_{j,k}$  depends on  $k$  with  $1 \leq k \leq q$ ,  $(k, q) = 1$ ,  $k \equiv a_j((q_j, q))$ . For such a  $k$  there exists an integer  $f_{j,k}$  such that the congruence  $m \equiv f_{j,k}([q_j, q])$  is equivalent to the system  $m \equiv a_j(q_j)$ ,  $m \equiv k(q)$ , so the last step follows.

Now for positive integers  $x$  and  $h \leq x$  let

$$\Delta(x, h) := \max_{y \leq x} \max_{\substack{l, h=1}} \left| \sum_{\substack{m \leq y \\ m \equiv l(h)}} \Lambda(m) - \frac{y}{\varphi(h)} \right|.$$

This expression is  $\geq 1$  for  $h \leq x$ . (Take  $y = \varphi(h)$  and  $l = 1$ ).

Note that by the Theorem of Bombieri and Vinogradov (see, for example, Brüdern [2]) we have

$$\sum_{h \leq U} \Delta(x, h) \ll \frac{x}{(\log x)^D} + U \sqrt{x} (\log(Ux))^6$$

for any fixed  $D \geq 1$ . This yields that if  $U \leq x^{1/2}/(\log x)^{D+6}$ , then

$$\sum_{h \leq U} \Delta(x, h) \ll \frac{x}{(\log x)^D}.$$

Now we compute  $T_{j,k}(\alpha)$  by partial summation and by introducing  $\Delta$ . We get

$$\begin{aligned} T_{j,k}(\alpha) &= \sum_{\substack{m \leq n \\ m \equiv f_{j,k}([q_j, q])}} \Lambda(m) e(\alpha m) \\ &= - \int_0^n \left( \sum_{\substack{m \leq y \\ m \equiv f_{j,k}([q_j, q])}} \Lambda(m) \right) \frac{d}{dy} (e(\alpha y)) dy + \left( \sum_{\substack{m \leq n \\ m \equiv f_{j,k}([q_j, q])}} \Lambda(m) \right) e(\alpha n) \\ &= - \int_0^n \left( \frac{y}{\varphi([q_j, q])} + O(\Delta(n, [q_j, q])) \right) \frac{d}{dy} e(\alpha y) dy \\ &\quad + \left( \frac{n}{\varphi([q_j, q])} + O(\Delta(n, [q_j, q])) \right) e(\alpha n) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\varphi([q_j, q])} \left( - \int_0^n y \left( \frac{d}{dy} e(\alpha y) \right) dy + n e(\alpha n) \right) + O\left( (1 + |\alpha|n) \Delta(n, [q_j, q]) \right) \\
&= \frac{1}{\varphi([q_j, q])} \int_0^n e(\alpha y) dy + O\left( \frac{R}{q} \Delta(n, [q_j, q]) \right),
\end{aligned}$$

since  $|\alpha| \leq \frac{R}{qn}$  and  $1 \leq \frac{R}{q}$ .

This yields, using

$$\int_0^n e(\alpha y) dy = M(\alpha) + O(1), \quad M(\alpha) := \sum_{m=1}^n e(\alpha m),$$

the expression

$$T_{j,k}(\alpha) = \frac{M(\alpha)}{\varphi([q_j, q])} + O\left( \frac{R}{q} \Delta(n, [q_j, q]) \right).$$

We use this term for  $T_{j,k}(\alpha)$  to compute  $S_j\left(\frac{a}{q} + \alpha\right)$  as

$$\begin{aligned}
S_j\left(\frac{a}{q} + \alpha\right) &= \sum_{\substack{1 \leq k \leq q \\ (k, q) = 1 \\ k \equiv a_j \pmod{(q_j, q)}}} e\left(\frac{a}{q}k\right) \left( \frac{M(\alpha)}{\varphi([q_j, q])} + O\left( \frac{R}{q} \Delta(n, [q_j, q]) \right) \right) + O((\log n)^2) \\
&= \frac{c_j(a, q)}{\varphi([q_j, q])} M(\alpha) + O\left( \frac{R}{q} \Delta(n, [q_j, q]) \right) + O((\log n)^2) \\
&= \frac{c_j(a, q)}{\varphi([q_j, q])} M(\alpha) + O\left( \frac{R}{q} (\log n)^2 \Delta(n, [q_j, q]) \right)
\end{aligned}$$

since  $(\log n)^2 \geq 1$  and  $\frac{R}{q} \Delta(n, [q_j, q]) \geq 1$ , with Ramanujan sums

$$c_j(a, q) := \sum_{\substack{1 \leq k \leq q \\ (k, q) = 1 \\ k \equiv a_j \pmod{(q, q_j)}}} e\left(\frac{a}{q}k\right) \text{ for } j = 1, 2, 3.$$

We used here that  $|c_j(a, q)| = 1$  or  $c_j(a, q) = 0$ , see paragraph 2.2.

This provides

$$I(a, q) = \int_{-\frac{R}{qn}}^{\frac{R}{qn}} S_1\left(\frac{a}{q} + \alpha\right) S_2\left(\frac{a}{q} + \alpha\right) S_3\left(\frac{a}{q} + \alpha\right) e\left(-n\left(\frac{a}{q} + \alpha\right)\right) d\alpha$$

$$= H_{a,q}(n) + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$$

with

$$\begin{aligned} H_{a,q}(n) &:= \frac{(c_1 c_2 c_3)(a, q)}{\varphi([q_1, q])\varphi([q_2, q])\varphi([q_3, q])} e\left(-n \frac{a}{q}\right) \int_{-\frac{R}{qn}}^{\frac{R}{qn}} M^3(\alpha) e(-n\alpha) d\alpha, \\ \mathcal{O}_1 &:= \sum_{j,k,l} \frac{1}{\varphi([q_j, q])\varphi([q_k, q])} \int_{-\frac{R}{qn}}^{\frac{R}{qn}} |M^2(\alpha)| d\alpha \cdot O\left(\frac{R}{q} (\log n)^2 \Delta(n, [q_l, q])\right), \\ \mathcal{O}_2 &:= \sum_{j,k,l} \frac{1}{\varphi([q_j, q])} \int_{-\frac{R}{qn}}^{\frac{R}{qn}} |M(\alpha)| d\alpha \cdot O\left(\frac{R^2}{q^2} (\log n)^4 \Delta(n, [q_k, q]) \Delta(n, [q_l, q])\right), \\ \mathcal{O}_3 &:= O\left(\frac{R^3}{q^3} (\log n)^6 \Delta(n, [q_1, q]) \Delta(n, [q_2, q]) \Delta(n, [q_3, q]) \frac{R}{qn}\right). \end{aligned}$$

Note that we abbreviated  $(c_1 c_2 c_3)(a, q) := c_1(a, q) c_2(a, q) c_3(a, q)$ . The sum  $\sum_{j,k,l}$  is over all triplets  $(j, k, l)$  of pairwise different  $j, k, l \in \{1, 2, 3\}$ .

So we managed to show

$$J_3^{\mathfrak{M}}(n) = \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} I(a, q) = \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} (H_{a,q}(n) + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3).$$

The main term of  $J_3^{\mathfrak{M}}(n)$  is contained in

$$H(n) := \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} H_{a,q}(n).$$

We have to show now that for each  $i = 1, 2, 3$  the error term  $\mathcal{O}_i$  fulfills

$$\sum_{q_1, q_2, q_3} \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \mathcal{O}_i \ll \frac{n^2}{(\log n)^{A+3}},$$

then it will follow that

$$\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} |J_3^{\mathfrak{M}}(n) - H(n)| \ll \frac{n^2}{(\log n)^{A+3}}.$$

The main term  $H(n)$  will be considered later.



So we first consider the error term with  $\mathcal{O}_1$ . It is (since  $\varphi(q) \gg q/(\log \log q)$ )

$$\begin{aligned}
&\ll \sum_{j,k,l} \sum_{q \leq R} \sum_{q_j, q_k} \frac{1}{\varphi([q_j, q])\varphi([q_k, q])} \sum_{\substack{a < q \\ (a,q)=1}} \frac{R^2}{q^2} n(\log n)^2 \sum_{q_l} \Delta(n, [q_l, q]) \\
&\ll \sum_{j,k,l} \sum_{q_j} \frac{\log \log n}{q_j} \sum_{q_k} \frac{\log \log n}{q_k} R^2 n(\log n)^2 \sum_{q_l} \sum_{q \leq R} \frac{1}{q} \Delta(n, [q_l, q]) \\
&\ll R^2 n(\log n)^5 \sum_{j,k,l} \sum_{h_l \leq RQ_l} \omega(h_l) \Delta(n, h_l)
\end{aligned}$$

with

$$\begin{aligned}
\omega(h_l) &:= \sum_{q_l} \sum_{\substack{q \leq R \\ [q_l, q] = h_l}} \frac{1}{q} = \sum_{d_l \leq R} \sum_{q_l} \sum_{\substack{q \leq R \\ (q_l, q) = d_l \\ q_l q = h_l d_l}} \frac{1}{q} \\
&\ll \sum_{d_l \leq R} \sum_{\substack{q \leq R \\ d_l | q}} \frac{1}{q} \ll \sum_{d_l \leq R} \sum_{q \leq R} \frac{1}{qd_l} \ll (\log n)^2,
\end{aligned}$$

so the  $\mathcal{O}_1$ -error term is

$$\begin{aligned}
&\ll R^2 n(\log n)^7 \sum_{j,k,l} \sum_{h \leq RQ_l} \Delta(n, h) \ll R^2 n(\log n)^7 \frac{n}{(\log n)^D} \\
&\ll \frac{n^2}{(\log n)^{D-2B-7}} \ll \frac{n^2}{(\log n)^{A+3}},
\end{aligned}$$

for some  $D \geq A+2B+10$  and  $D+6 \leq \vartheta - B$ , so this holds if  $\vartheta \geq A+3B+16$ , which is the case. We used the Theorem of Bombieri and Vinogradov with  $Q_l \ll n^{1/2}(\log n)^{-\vartheta}$  for  $\vartheta > 0$ . So we are done for  $\mathcal{O}_1$ .

We consider now the error term with  $\mathcal{O}_2$ . It is

$$\begin{aligned}
&\ll \sum_{j,k,l} \sum_{q \leq R} \sum_{q_j} \frac{1}{\varphi([q_j, q])} \sum_{\substack{a < q \\ (a,q)=1}} \frac{R^3}{q^3} (\log n)^4 \sum_{q_k, q_l} \Delta(n, [q_k, q]) \Delta(n, [q_l, q]) \\
&\ll \sum_{j,k,l} \sum_{q_j} \frac{\log \log n}{q_j} R^3 (\log n)^4 \sum_{q_k, q_l} \sum_{q \leq R} \frac{1}{q^2} \Delta(n, [q_k, q]) \Delta(n, [q_l, q]) \\
&\ll R^3 (\log n)^6 \sum_{j,k,l} \sum_{h_k \leq RQ_k} \sum_{h_l \leq RQ_l} \omega(h_k, h_l) \Delta(n, h_k) \Delta(n, h_l)
\end{aligned}$$

with

$$\begin{aligned}
\omega(h_k, h_l) &:= \sum_{q_k, q_l} \sum_{\substack{q \leq R \\ [q_k, q] = h_k \\ [q_l, q] = h_l}} \frac{1}{q^2} = \sum_{d_k, d_l \leq R} \sum_{q_k, q_l} \sum_{\substack{q \leq R \\ (q_k, q) = d_k, (q_l, q) = d_l \\ q_k q = h_k d_k, q_l q = h_l d_l}} \frac{1}{q^2} \\
&\ll \sum_{d_k, d_l \leq R} \sum_{\substack{q \leq R \\ [d_k, d_l] | q}} \frac{1}{q^2} \leq \sum_{d_k, d_l \leq R} \sum_{q \leq R} \frac{1}{q^2 [d_k, d_l]^2} \\
&= \sum_{d_k, d_l \leq R} \sum_{q \leq R} \frac{(d_k, d_l)^2}{q^2 d_k^2 d_l^2} \leq R^2 \sum_q \frac{1}{q^2} \sum_{d_k} \frac{1}{d_k^2} \sum_{d_l} \frac{1}{d_l^2} \ll R^2,
\end{aligned}$$

so the  $\mathcal{O}_2$ -error term is

$$\begin{aligned}
&\ll R^5 (\log n)^6 \sum_{h_k \leq RQ_k} \Delta(n, h_k) \sum_{h_l \leq RQ_l} \Delta(n, h_l) \\
&\ll R^5 (\log n)^6 \cdot \left( \frac{n}{(\log n)^D} \right)^2 = \frac{n^2}{(\log n)^{2D-5B-6}} \ll \frac{n^2}{(\log n)^{A+3}},
\end{aligned}$$

for some  $2D \geq A+5B+9$  and  $D+6 \leq \vartheta-B$ , so this holds if  $\vartheta \geq \frac{1}{2}A + \frac{7}{2}B + 11$ , which is the case. We used the Theorem of Bombieri and Vinogradov with  $Q_k, Q_l \ll n^{1/2}(\log n)^{-\vartheta}$  for  $\vartheta > 0$ . So we are done for  $\mathcal{O}_2$ .

Now to the error term with  $\mathcal{O}_3$ , it is

$$\begin{aligned}
&\ll \sum_{q \leq R} \sum_{\substack{a \leq q \\ (a, q) = 1}} \frac{R^4}{q^4 n} (\log n)^6 \sum_{q_1, q_2, q_3} \Delta(n, [q_1, q]) \Delta(n, [q_2, q]) \Delta(n, [q_3, q]) \\
&\ll \frac{R^4}{n} (\log n)^6 \sum_{\substack{h_1 \leq RQ_1 \\ h_2 \leq RQ_2 \\ h_3 \leq RQ_3}} \omega(h_1, h_2, h_3) \Delta(n, h_1) \Delta(n, h_2) \Delta(n, h_3)
\end{aligned}$$

with

$$\begin{aligned}
\omega(h_1, h_2, h_3) &:= \sum_{q_1, q_2, q_3} \sum_{\substack{q \leq R \\ [q_i, q] = h_i}} \frac{1}{q^3} = \sum_{d_1, d_2, d_3 \leq R} \sum_{q_1, q_2, q_3} \sum_{\substack{q \leq R \\ (q_i, q) = d_i \\ q_i q = h_i d_i}} \frac{1}{q^3} \ll \sum_{d_1, d_2, d_3 \leq R} \sum_{\substack{q \leq R \\ [d_1, d_2, d_3] | q}} \frac{1}{q^3} \\
&\ll \sum_{d_1, d_2, d_3 \leq R} \sum_{q \leq R} \frac{1}{q^3 [d_1, d_2, d_3]^3} = \sum_{d_1, d_2, d_3 \leq R} \sum_{q \leq R} \frac{(d_1, [d_2, d_3])^3 (d_2, d_3)^3}{q^3 d_1^3 d_2^3 d_3^3}
\end{aligned}$$

$$\ll \sum_{d_1, d_2 \leq R} \sum_{d_3 \leq R} \frac{1}{d_3^3} \sum_{q \leq R} \frac{1}{q^3} \ll R^2,$$

so the  $\mathcal{O}_3$ -error term is

$$\begin{aligned} &\ll \frac{R^6}{n} (\log n)^6 \sum_{h_1 \leq RQ_1} \Delta(n, h_1) \sum_{h_2 \leq RQ_2} \Delta(n, h_2) \sum_{h_3 \leq RQ_3} \Delta(n, h_3) \\ &\ll \frac{R^6}{n} (\log n)^6 \frac{n^3}{(\log n)^{3D}} = \frac{n^2}{(\log n)^{3D-6B-6}} \ll \frac{n^2}{(\log n)^A}, \end{aligned}$$

for some  $3D \geq A+6B+9$  and  $D+6 \leq \vartheta - B$ , so this holds if  $\vartheta \geq \frac{1}{3}A+3B+9$ , which is the case. We used the Theorem of Bombieri and Vinogradov with  $Q_1, Q_2, Q_3 \ll n^{1/2}(\log n)^{-\vartheta}$  for  $\vartheta > 0$ . So we are done for  $\mathcal{O}_3$ .

What is now left is the consideration of the main term  $H(n)$ . Since

$$\int_{-\frac{R}{qn}}^{\frac{R}{qn}} M^3(\alpha) e(-n\alpha) d\alpha = \frac{n^2}{2} + O\left(\frac{q^2 n^2}{R^2}\right)$$

(see for example Vaughan [5]) we have

$$H(n) = \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \frac{(c_1 c_2 c_3)(a, q) e\left(-n \frac{a}{q}\right)}{\varphi([q_1, q]) \varphi([q_2, q]) \varphi([q_3, q])} \left( \frac{n^2}{2} + O\left(\frac{q^2 n^2}{R^2}\right) \right).$$

Now let

$$\lambda(q) := \frac{\varphi(q_1) \varphi(q_2) \varphi(q_3)}{\varphi([q_1, q]) \varphi([q_2, q]) \varphi([q_3, q])} b(q)$$

with

$$b(q) := \sum_{\substack{a < q \\ (a, q) = 1}} (c_1 c_2 c_3)(a, q) e\left(-n \frac{a}{q}\right)$$

and let

$$\mathcal{S}_3(n) := \sum_{q=1}^{\infty} \lambda(q)$$

be the singular series. In the next paragraph we show that it is absolutely convergent.

Therefore we have

$$\begin{aligned} H(n) &= \sum_{q \leq R} \frac{\lambda(q)n^2}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} + O\left(\frac{n^2}{R^2} \sum_{q \leq R} \frac{q^2|\lambda(q)|}{\varphi(q_1)\varphi(q_2)\varphi(q_3)}\right) \\ &= \frac{n^2}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \mathcal{S}_3(n) + O(e_1) + O(e_2) \end{aligned}$$

with

$$\begin{aligned} e_1 &:= \frac{n^2}{\varphi(q_1)\varphi(q_2)\varphi(q_3)} \sum_{q > R} |\lambda(q)|, \\ e_2 &:= \frac{n^2}{R^2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \sum_{q \leq R} q^2 |\lambda(q)|. \end{aligned}$$

For the two occuring error terms  $e_1$  and  $e_2$  we have to show that

$$\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} e_j \ll \frac{n^2}{(\log n)^{A+3}},$$

then Theorem 3 follows. This is done in the next paragraph.

## 2.2 Estimations with the singular series

Now we need estimations for the  $\lambda$ -series. These show the absolute convergence of  $\mathcal{S}_3(n)$  and can also be used to deal with  $e_1$  and  $e_2$ .

First we state that the Ramanujan sums  $c_j(a, q)$  for fixed  $a_j, q_j$ ,  $j = 1, 2, 3$ , can be computed by

$$c_j(a, q) = c_{a_j, q_j}(a, q) = \begin{cases} \mu\left(\frac{q}{d_j}\right) e\left(\frac{au_j a_j}{d_j}\right), & \text{if } (d_j, \frac{q}{d_j}) = 1, \\ 0, & \text{else,} \end{cases}$$

where  $d_j := (q_j, q)$  and  $u_j$  is the solution of the congruence  $\frac{q}{d_j} u_j \equiv 1 \pmod{d_j}$ , with  $0 \leq u_j < d_j$ . (For a proof see [6]). From this result we already used that  $|c_j(a, q)| = 1$  or  $c_j(a, q) = 0$  in the paragraph before.

We are now going to show that  $b$  is multiplicative in  $q$ . We prove a proposition about the  $c_j$  first.

**Proposition 1.** Let  $q = \bar{q}\tilde{q}$ ,  $(\bar{q}, \tilde{q}) = 1$ ,  $(a, q) = 1$ , and let  $a = \tilde{a}\bar{q} + \bar{a}\tilde{q}$  with  $(\tilde{a}, \tilde{q}) = 1$ ,  $(\bar{a}, \bar{q}) = 1$ . Then  $c_j(a, q) = c_j(\tilde{a}, \tilde{q}) \cdot c_j(\bar{a}, \bar{q})$  for  $j = 1, 2, 3$ .

**Proof.** Let  $\tilde{a}_j\bar{q} + \bar{a}_j\tilde{q} \equiv a_j \pmod{(q_j, q)}$  with  $\tilde{a}_j$  a residue mod  $(q_j, \tilde{q})$  and  $\bar{a}_j$  a residue mod  $(q_j, \bar{q})$ . Then we have for  $j = 1, 2, 3$

$$c_j(a, q) = \sum_{\substack{m < q \\ (m, q) = 1 \\ m \equiv a_j \pmod{(q_j, q)}}} e\left(m \frac{a}{q}\right) = \sum_{\substack{\tilde{m} < \tilde{q} \\ (\tilde{m}, \tilde{q}) = 1 \\ \tilde{m} \equiv \tilde{a}_j \pmod{(q_j, \tilde{q})}}} \sum_{\substack{\bar{m} < \bar{q} \\ (\bar{m}, \bar{q}) = 1 \\ \bar{m} \equiv \bar{a}_j \pmod{(q_j, \bar{q})}}} e\left(\frac{(\tilde{m}\bar{q} + \bar{m}\tilde{q})(\tilde{a}\bar{q} + \bar{a}\tilde{q})}{\tilde{q}\bar{q}}\right)$$

by substituting  $m = \tilde{m}\bar{q} + \bar{m}\tilde{q}$  with  $\tilde{m} \equiv \tilde{a}_j \pmod{(q_j, \tilde{q})}$  and  $\bar{m} \equiv \bar{a}_j \pmod{(q_j, \bar{q})}$ , and we have  $\tilde{a}_j \equiv a_j \bar{q}^{-1} \pmod{(q_j, \tilde{q})}$  and  $\bar{a}_j \equiv a_j \tilde{q}^{-1} \pmod{(q_j, \bar{q})}$ . Therefore we get

$$\begin{aligned} c_j(a, q) &= \sum_{\substack{\tilde{m} < \tilde{q} \\ (\tilde{m}, \tilde{q}) = 1 \\ \tilde{m} \equiv a_j \bar{q}^{-1} \pmod{(q_j, \tilde{q})}}} e\left(\frac{\tilde{m}\tilde{a}\tilde{q}}{\tilde{q}}\right) \sum_{\substack{\bar{m} < \bar{q} \\ (\bar{m}, \bar{q}) = 1 \\ \bar{m} \equiv a_j \tilde{q}^{-1} \pmod{(q_j, \bar{q})}}} e\left(\frac{\bar{m}\bar{a}\tilde{q}}{\bar{q}}\right) \\ &= \sum_{\substack{\tilde{m} < \tilde{q} \\ (\tilde{m}, \tilde{q}) = 1 \\ \tilde{m} \equiv a_j \pmod{(q_j, \tilde{q})}}} e\left(\frac{\tilde{m}\tilde{a}}{\tilde{q}}\right) \sum_{\substack{\bar{m} < \bar{q} \\ (\bar{m}, \bar{q}) = 1 \\ \bar{m} \equiv a_j \pmod{(q_j, \bar{q})}}} e\left(\frac{\bar{m}\bar{a}}{\bar{q}}\right) \\ &= c_j(\tilde{a}, \tilde{q}) \cdot c_j(\bar{a}, \bar{q}). \end{aligned} \quad \square$$

Proposition 1 provides the multiplicativity of  $b$ :

**Proposition 2.** Let  $(\bar{q}, \tilde{q}) = 1$ . Then  $b(\bar{q}\tilde{q}) = b(\bar{q})b(\tilde{q})$ .

**Proof.** We have

$$\begin{aligned} b(\bar{q}\tilde{q}) &= \sum_{\substack{a < \bar{q}\tilde{q} \\ (a, \bar{q}\tilde{q}) = 1}} (c_1 c_2 c_3)(a, \bar{q}\tilde{q}) e\left(-n \frac{a}{\bar{q}\tilde{q}}\right) \\ &= \sum_{\substack{\bar{a} < \bar{q} \\ (\bar{a}, \bar{q}) = 1}} \sum_{\substack{\tilde{a} < \tilde{q} \\ (\tilde{a}, \tilde{q}) = 1}} (c_1 c_2 c_3)(\tilde{a}, \tilde{q}) \cdot (c_1 c_2 c_3)(\bar{a}, \bar{q}) \cdot e\left(-n \frac{\tilde{a}\bar{q} + \bar{a}\tilde{q}}{\bar{q}\tilde{q}}\right) \end{aligned}$$

by substituting  $a = \tilde{a}\bar{q} + \bar{a}\tilde{q}$  in the last step. We further get

$$b(\bar{q}\tilde{q}) = \sum_{\substack{\bar{a} < \bar{q} \\ (\bar{a}, \bar{q}) = 1}} (c_1 c_2 c_3)(\bar{a}, \bar{q}) e\left(-n \frac{\bar{a}}{\bar{q}}\right) \sum_{\substack{\tilde{a} < \tilde{q} \\ (\tilde{a}, \tilde{q}) = 1}} (c_1 c_2 c_3)(\tilde{a}, \tilde{q}) e\left(-n \frac{\tilde{a}}{\tilde{q}}\right)$$

$$= b(\bar{q}) \cdot b(\tilde{q}). \quad \square$$

Proposition 2 shows that it suffices to evaluate  $b$  at prime powers  $p^k$ ,  $p$  prime and  $k \geq 1$ , to obtain formulas for  $b$  and  $\lambda$ . It may happen that  $b(p^k) = 0$ , what we study now.

We first show:

**Proposition 3.** *Let  $j \in \{1, 2, 3\}$ . If  $p^k \nmid q_j$  and  $(p \mid q_j \text{ or } k \neq 1)$ , then  $c_j(a, p^k) = 0$ .*

**Proof.**

Firstly, if  $p^k \nmid q_j$  and  $p \mid q_j$ , we have  $d_j = (q_j, p^k) = p^r$  with  $1 \leq r < k$  and  $(d_j, \frac{p^k}{d_j}) = (p^r, p^{k-r}) \geq p$ , so  $c_j(a, p^k) = 0$ .

Secondly, if  $p^k \nmid q_j$  and  $k \neq 1$ , then  $d_j = (q_j, p^k) = p^r$  with  $0 \leq r < k$ , and

$$\left(d_j, \frac{p^k}{d_j}\right) = (p^r, p^{k-r}) = p^{\min(r, k-r)}.$$

For  $r > 0$  this is  $\geq p$ , and so  $c_j(a, p^k) = 0$ . For  $r = 0$  we have  $d_j = 1$  and  $\mu(\frac{p^k}{d_j}) = \mu(p^k) = 0$  since  $k \neq 1$ , so  $c_j(a, p^k) = 0$ , too.  $\square$

Therefore  $c_j(a, p^k) = 0$  holds unless  $p^k \mid q_j$  or  $(p \nmid q_j \text{ and } k = 1)$ . This shows that

$$b(p^k) = \sum_{\substack{a < p^k \\ (a, p) = 1}} c_1(a, p^k) c_2(a, p^k) c_3(a, p^k) e\left(-n \frac{a}{p^k}\right) = 0,$$

unless  $p^k \mid q_j$  or  $(p \nmid q_j \text{ and } k = 1)$  for every  $j = 1, 2, 3$ . We now have to consider only these cases.

**Case 1.** If  $k \geq 1$ ,  $p^k \mid (q_1, q_2, q_3)$ , then

$$\begin{aligned} b(p^k) &= \sum_{\substack{a < p^k \\ (a, p) = 1}} c_1(a, p^k) c_2(a, p^k) c_3(a, p^k) e\left(-n \frac{a}{p^k}\right) \\ &= \sum_{\substack{a < p^k \\ (a, p) = 1}} e\left(-n \frac{a}{p^k}\right) \prod_{i=1,2,3} e\left(\frac{a q_i}{p^k}\right) \quad (\text{since } d_i = (q_i, p^k) = p^k \text{ so } u_i = 1) \end{aligned}$$

$$= \sum_{\substack{a < p^k \\ (a,p)=1}} e \left( \frac{a_1 + a_2 + a_3 - n}{p^k} a \right),$$

so  $b(p^k) = \varphi(p^k)$  since  $p^k \mid a_1 + a_2 + a_3 - n$  by the general condition.

**Case 2.** If  $k = 1$  and  $(p, q_1) = (p, q_2) = (p, q_3) = 1$  then

$$\begin{aligned} b(p) &= \sum_{a=1}^{p-1} c_1(a, p) c_2(a, p) c_3(a, p) e \left( -n \frac{a}{p} \right) \\ &= \sum_{a=1}^{p-1} e \left( -n \frac{a}{p} \right) \prod_{i=1,2,3} \underbrace{\sum_{m=1}^{p-1} e \left( m \frac{a}{p} \right)}_{=-1} = \begin{cases} 1 - p, & \text{if } p \mid n, \\ 1, & \text{if } p \nmid n. \end{cases} \end{aligned} \quad \begin{matrix} (A) \\ (B) \end{matrix}$$

**Case 3.** If  $k = 1$ ,  $p \mid q_1$  (so  $d_1 = p$ ) and  $(p, q_2) = (p, q_3) = 1$  (analogously the cases with permuted indices), then

$$\begin{aligned} b(p) &= \sum_{a=1}^{p-1} c_1(a, p) c_2(a, p) c_3(a, p) e \left( -n \frac{a}{p} \right) \\ &= \sum_{a=1}^{p-1} e \left( -n \frac{a}{p} \right) e \left( \frac{aa_1}{p} \right) \underbrace{\left( \sum_{m=1}^{p-1} e \left( m \frac{a}{p} \right) \right)^2}_{=-1} \\ &= \sum_{a=1}^{p-1} e \left( \frac{a_1 - n}{p} a \right) = \begin{cases} p - 1, & \text{if } p \mid a_1 - n, \\ -1, & \text{if } p \nmid a_1 - n. \end{cases} \end{aligned} \quad \begin{matrix} (C) \\ (D) \end{matrix}$$

**Case 4.** If  $k = 1$ ,  $p \mid q_1$ ,  $p \mid q_2$  and  $(p, q_3) = 1$  (analogously the cases with permuted indices), then

$$\begin{aligned} b(p) &= \sum_{a=1}^{p-1} c_1(a, p) c_2(a, p) c_3(a, p) e \left( -n \frac{a}{p} \right) \\ &= \sum_{a=1}^{p-1} e \left( -n \frac{a}{p} \right) e \left( \frac{aa_1}{p} \right) e \left( \frac{aa_2}{p} \right) \underbrace{\sum_{m=1}^{p-1} e \left( m \frac{a}{p} \right)}_{=-1} \end{aligned}$$

$$= - \sum_{a=1}^{p-1} e \left( \frac{a_1 + a_2 - n}{p} a \right) = \begin{cases} 1 - p, & \text{if } p \mid a_1 + a_2 - n, \\ 1, & \text{if } p \nmid a_1 + a_2 - n. \end{cases} \quad \begin{matrix} (E) \\ (F) \end{matrix}$$

If we combine all these cases, we have shown

1. If  $k \geq 1$  and  $p^k \mid (q_1, q_2, q_3)$ :  $b(p^k) = \varphi(p^k)$ , furthermore  $\lambda(p^k) = b(p^k)$ .
2. If  $p \nmid (q_1, q_2, q_3)$ :

$$b(p) = \begin{cases} 1 - p, & \text{if } (p, q_1) = (p, q_2) = (p, q_3) = 1, \ p \mid n, & (A) \\ 1, & \text{if } (p, q_1) = (p, q_2) = (p, q_3) = 1, \ p \nmid n, & (B) \\ p - 1, & \text{if } p \mid q_1, \ (p, q_2) = (p, q_3) = 1, \ p \mid a_1 - n, & (C) \\ & \text{also with permuted indices,} \\ -1, & \text{if } p \mid q_1, \ (p, q_2) = (p, q_3) = 1, \ p \nmid a_1 - n, & (D) \\ & \text{also with permuted indices,} \\ 1 - p, & \text{if } p \mid q_1, \ p \mid q_2, \ (p, q_3) = 1, \ p \mid a_1 + a_2 - n, & (E) \\ & \text{also with permuted indices,} \\ 1, & \text{if } p \mid q_1, \ p \mid q_2, \ (p, q_3) = 1, \ p \nmid a_1 + a_2 - n, & (F) \\ & \text{also with permuted indices,} \end{cases}$$

so  $b(p) \in \{\pm 1, \pm(p-1)\}$ . Expressed in  $\lambda$  we have

$$\lambda(p) = \begin{cases} \frac{1}{(p-1)^3}, & (B) \\ -\frac{1}{(p-1)^2}, & (A), (D) \\ \frac{1}{p-1}, & (C), (F) \\ -1. & (E) \end{cases}$$

3. In any other case:  $b(p^k) = \lambda(p^k) = 0$ .

In the following let  $d := (q_1, q_2, q_3)$  where the  $q_j$  are fixed. For a prime  $p$  let  $\gamma_p$  such that  $p^{\gamma_p} \parallel d$ , that is  $p^{\gamma_p} \mid d$  but  $p^{\gamma_p+1} \nmid d$ .

Now with  $b$ ,  $\lambda$  is multiplicative too, since  $\varphi(q_i)\varphi([q_i, \bar{q}\tilde{q}]) = \varphi([q_i, \bar{q}])\varphi([q_i, \tilde{q}])$  for  $(\bar{q}, \tilde{q}) = 1$ ,  $i = 1, 2, 3$ . This multiplicativity for  $\lambda$  shows for  $Q \geq 1$ :

$$\sum_{q=1}^Q q|\lambda(q)| \leq \prod_{\substack{p \leq Q \\ \text{prime}}} \left( 1 + \sum_{k=1}^{2 \log Q} p^k |\lambda(p^k)| \right)$$



$$\begin{aligned}
&= \left( \prod_{p \leq Q, p|d} (1 + p|\lambda(p)| + p^2|\lambda(p^2)| + \cdots + p^{\gamma_p}|\lambda(p^{\gamma_p})|) \right) \cdot \left( \prod_{p \leq Q, (p,d)=1} (1 + p|\lambda(p)|) \right) \\
&\leq \left( \prod_{p|d} (1 + p(p-1) + p^2(p^2-p) + \cdots + p^{\gamma_p}(p^{\gamma_p} - p^{\gamma_p-1})) \right) \cdot \left( \prod_{p \leq Q, (p,d)=1} (1 + p|\lambda(p)|) \right) \\
&\leq \left( \prod_{p|d} p^{2\gamma_p} \right) \cdot \left( \prod_{p \leq Q, (p,d)=1} (1 + p|\lambda(p)|) \right) = d^2 \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{D},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A} &:= \prod_{p \leq Q, (B)} \left( 1 + \frac{p}{\varphi(p)^3} \right) \leq \sum_{q, p|q \Rightarrow p \leq Q} \frac{q\mu^2(q)}{\varphi(q)^3} \ll \sum_q \frac{\mu^2(q)}{q^2} (\log \log q)^3 \ll 1, \\
\mathcal{B} &:= \prod_{p \leq Q, (A)} \left( 1 + \frac{p}{\varphi(p)^2} \right) \cdot \prod_{i=1,2,3} \prod_{\substack{p \leq Q, \\ (D) \text{ for } q_i}} \left( 1 + \frac{p}{\varphi(p)^2} \right) \\
&\leq \left( \sum_{\substack{q \leq n \\ p|q \Rightarrow p|n}} \frac{q\mu^2(q)}{\varphi(q)^2} \right) \prod_{i=1,2,3} \left( \sum_{\substack{q \leq q_i \\ p|q \Rightarrow p|q_i}} \frac{q\mu^2(q)}{\varphi(q)^2} \right) \ll \left( \sum_{q \leq n} \frac{\mu^2(q)}{q} (\log \log q)^2 \right)^4 \\
&\ll (\log n)^8, \\
\mathcal{C} &:= \prod_{i=1,2,3} \prod_{\substack{p \leq Q \\ (C) \text{ for } q_i}} \left( 1 + \frac{p}{p-1} \right) \cdot \prod_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} \prod_{\substack{p \leq Q \\ (F) \text{ for } q_i, q_j}} \left( 1 + \frac{p}{p-1} \right) \\
&\leq \prod_{i=1,2,3} \prod_{p|q_i} (1+2) \cdot \prod_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} \prod_{p|(q_i, q_j)} (1+2) \leq \prod_{i=1,2,3} 2^{2\omega(q_i)} \cdot \prod_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} 2^{2\omega((q_i, q_j))} \\
&\leq \tau^2(q_1) \tau^2(q_2) \tau^2(q_3) \tau^2((q_1, q_2)) \tau^2((q_1, q_3)) \tau^2((q_2, q_3)) \leq \tau^4(q_1) \tau^4(q_2) \tau^4(q_3), \\
\mathcal{D} &:= \prod_{\substack{i,j,k \in \{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \prod_{\substack{p \leq Q \\ (E) \text{ for } q_i, q_j, q_k}} (1+p) \leq \prod_{\substack{i,j,k \in \{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \prod_{\substack{p|(q_i, q_j) \\ p \nmid q_k}} (1+p) \\
&= \prod_{\substack{i,j,k \in \{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \sigma \left( \prod_{\substack{p|(q_i, q_j) \\ p \nmid q_k}} p \right) \leq \prod_{\substack{i,j,k \in \{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \sigma \left( \frac{(q_i, q_j)}{(q_1, q_2, q_3)} \right) \ll \prod_{\substack{i,j,k \in \{1,2,3\} \\ i,j,k \text{ p.w.d.}}} \frac{(q_i, q_j)}{d} \log n \\
&= \frac{1}{d^3} (q_1, q_2) (q_1, q_3) (q_2, q_3) (\log n)^3,
\end{aligned}$$

where  $\sigma(t) := \sum_{t|q} t$  is the divisor sum function, for which  $\sigma(t) \ll t \log t$  holds, and  $\omega(t)$  is the number of distinct prime factors of  $t$ .

Therefore

$$\sum_{q=1}^Q q|\lambda(q)| \ll \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{(q_1, q_2, q_3)} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (\log n)^{11},$$

also true for  $Q \rightarrow \infty$ . So for any  $Q \geq 1$  we have

$$\sum_{q \geq Q} |\lambda(q)| \leq \frac{1}{Q} \sum_{q=1}^{\infty} q|\lambda(q)| \leq \frac{1}{Q} \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{(q_1, q_2, q_3)} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (\log n)^{11}.$$

We see that the singular series  $\mathcal{S}_3(n) = \sum_{q=1}^{\infty} \lambda(q)$  converges absolutely, and we have

$$\mathcal{S}_3(n) \ll \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{(q_1, q_2, q_3)} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (\log n)^{11}.$$

It follows further that

$$\begin{aligned} \sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} e_1 &\ll \sum_{q_1, q_2, q_3} \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{d\varphi(q_1)\varphi(q_2)\varphi(q_3)} \frac{n^2}{R} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (\log n)^{11} \\ &\ll \frac{n^2}{R} (\log n)^{12} \sum_{q_1, q_2, q_3} \frac{\tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (q_1, q_2)(q_1, q_3)(q_2, q_3)}{q_1 q_2 q_3 d} \\ &\ll \frac{n^2}{R} (\log n)^{\eta} \ll \frac{n^2}{(\log n)^{A+3}}, \end{aligned}$$

since  $B \geq A + \eta + 3$  in  $R = (\log n)^B$  for some absolute constant  $\eta > 0$ . This can be proven as follows. By using

$$\sum_{t \leq n} \frac{\tau^m(t)}{t} \leq (\log n)^{2^m},$$

we see that

$$\begin{aligned} &\sum_{q_1, q_2, q_3} \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{q_1 q_2 q_3 (q_1, q_2, q_3)} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) \\ &\leq \sum_{d \leq n} \sum_{a, b, c \leq n} \sum_{e, f, g \leq n} \frac{dadbdcc}{d^3 a^2 b^2 c^2 e f g d} \tau^{12}(d) \tau^8(a) \tau^8(b) \tau^8(c) \tau^4(e) \tau^4(f) \tau^4(g) \end{aligned}$$

$$\begin{aligned}
&= \sum_d \sum_{a,b,c} \sum_{e,f,g} \frac{\tau^{12}(d)\tau^8(a)\tau^8(b)\tau^8(c)\tau^4(e)\tau^4(f)\tau^4(g)}{abcdefg} \\
&\ll (\log n)^\eta
\end{aligned}$$

for some absolute constant  $\eta > 0$ , where we substituted  $q_1 = dabe$ ,  $q_2 = dacf$ ,  $q_3 = dcbg$  with pairwise coprime  $a, b, c$  and  $e, f, g$ .

Further we have

$$\sum_{q \leq R} q^2 |\lambda(q)| \leq R \sum_{q=1}^{\infty} q |\lambda(q)| \ll R \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) \frac{(q_1, q_2)(q_1, q_3)(q_2, q_3)}{d} (\log n)^{11},$$

so also

$$\begin{aligned}
\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} e_2 &\ll \sum_{q_1, q_2, q_3} \frac{n^2 (q_1, q_2)(q_1, q_3)(q_2, q_3)}{R \varphi(q_1) \varphi(q_2) \varphi(q_3) d} \tau^4(q_1) \tau^4(q_2) \tau^4(q_3) (\log n)^{11} \\
&\ll \frac{n^2}{(\log n)^{A+3}}
\end{aligned}$$

as above.

So everything concerning Theorem 3 is shown.  $\square$

### 2.3 Discussion of the singular series

Now we consider  $\mathcal{S}_3(n)$  under the general condition.

Since  $\mathcal{S}_3(n)$  is absolutely convergent and since  $\lambda$  is multiplicative, we see that it has an Eulerproduct, namely

$$\mathcal{S}_3(n) = \prod_p \left( 1 + \sum_{k=1}^{\infty} \lambda(p^k) \right).$$

For  $p^\alpha \parallel (q_1, q_2, q_3)$  we have  $1 + \lambda(p) + \dots + \lambda(p^\alpha) = p^\alpha$  and for other primes  $p$  we get factors according to the cases (A), ..., (F). Moreover we see that  $\mathcal{S}_3(n)$  vanishes if case (E) for a prime  $p$  occurs, that is if

$$(E) : \quad \exists j, k, l \in \{1, 2, 3\} \text{ pairwise different with}$$

$$p \mid (q_j, q_k), p \nmid q_l, p \mid n - (a_j + a_k).$$

In all other cases we have

$$\mathcal{S}_3(n) = (q_1, q_2, q_3) \prod_{\substack{p, (A) \\ \text{or } (D)}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p, (B)} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{\substack{p, (C) \\ \text{or } (F)}} \left(1 + \frac{1}{p-1}\right)$$

with properties

$$\begin{aligned} (A) : & (p, q_1) = (p, q_2) = (p, q_3) = 1, p \mid n, \\ (B) : & (p, q_1) = (p, q_2) = (p, q_3) = 1, p \nmid n, \\ (C) : & \exists j, k, l \in \{1, 2, 3\} \text{ p.w.d. } p \mid q_j, (p, q_k) = (p, q_l) = 1, p \mid n - a_j, \\ (D) : & \exists j, k, l \in \{1, 2, 3\} \text{ p.w.d. } p \mid q_j, (p, q_k) = (p, q_l) = 1, p \nmid n - a_j, \\ (F) : & \exists j, k, l \in \{1, 2, 3\} \text{ p.w.d. } p \mid q_j, p \mid q_k, (p, q_l) = 1, p \nmid n - (a_j + a_k). \end{aligned}$$

So we see that  $\mathcal{S}_3(n) = 0$  if and only if case (E) occurs or the general condition is not fulfilled. Further if  $\mathcal{S}_3(n) > 0$  we see from the Eulerproduct that it is at least some absolute positive constant times  $(q_1, q_2, q_3)$ , since  $\prod_{p>2} (1 - (p-1)^{-2})$  converges and the other products are  $> 1$ .

Now we prove

**Lemma 1.** *If  $n$  is odd, then for given  $q_3, a_3$  with  $(a_3, q_3) = 1$  and  $q_2$  there exists an admissible  $a_2$  (such that for every  $q_1$  there exists an admissible  $a_1$ ). For even  $n$  and given  $q_1, q_2, q_3$  there exists no admissible triplet  $a_1, a_2, a_3$ .*

Recall that  $a_1, a_2, a_3$  is admissible for  $q_1, q_2, q_3$ , if  $(a_i, q_i) = 1$  for  $i = 1, 2, 3$ ,  $n \equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)}$  and  $\mathcal{S}_3(n) > 0$ .

**Proof.** For the proof, let  $q := (q_1, q_2, q_3)$  and denote by  $\nu_p(m)$  the exponent of a prime  $p$  in  $m$ , that is  $p^{\nu_p(m)} \mid m$  but  $p^{\nu_p(m)+1} \nmid m$ .

First let  $n$  be even, and consider  $q_1, q_2, q_3$  with

- (a)  $2 \mid q_j, 2 \nmid q_k, q_l$ . Then (A), (B), (E), (F) are not possible, and the condition  $2 \mid n - a_j$  is wrong since  $a_j$  must be odd. Therefore (D) holds with  $p = 2$ , and so  $\mathcal{S}_3(n) = 0$ .
- (b)  $2 \mid q_j, q_k$  and  $2 \nmid q_l$ . Then (A),  $\dots$ , (D) are not possible, and condition  $2 \mid n - (a_j + a_k)$  in (E) holds since  $a_j, a_k$  are odd, so  $\mathcal{S}_3(n) = 0$ .
- (c) Further  $2 \mid q_1, q_2, q_3$  is not possible since then  $a_1, a_2, a_3$  are odd and so  $n \not\equiv a_1 + a_2 + a_3 \pmod{q}$ , so  $\mathcal{S}_3(n) = 0$ .

(d) Also  $2 \nmid q_1, q_2, q_3$  is not possible since then (A) holds for  $p = 2$ , so  $\mathcal{S}_3(n) = 0$  holds.

Now let  $n$  be odd and let  $q_3, a_3$  with  $(a_3, q_3) = 1$  and  $q_2$  be given. We construct  $a_2$  and  $q_2$  with  $(a_2, q_2) = 1$  such that

$$\forall p \mid (q_3, q_2) : n \not\equiv a_3 + a_2 \pmod{p}.$$

For any  $p \mid (q_3, q_2)$  take  $h_p$  such that  $1 \leq h_p \leq p-1$  with  $n - a_3 + h_p \not\equiv 0 \pmod{p}$ . Such a number  $h_p$  exists for  $p > 2$  since then  $p-1 > 1$ , and if  $p = 2$  take  $h_2 = 1$  since  $n - a_3 + 1 \not\equiv 0 \pmod{2}$  holds for  $p = 2 \mid (q_3, q_2)$ , where  $q_3$  is even and therefore  $a_3$  is odd.

Then take  $a_2$  with  $(a_2, q_2) = 1$  and  $a_2 \equiv n - a_3 + h_p \pmod{p}$  for every  $p \mid (q_3, q_2)$  via the Chinese Remainder Theorem. Now we prove that this  $a_2$  is admissible. For this, consider now any  $q_1$ , and we have to find now an admissible  $a_1$ , that means such that

- (1)  $n \equiv a_1 + a_2 + a_3 \pmod{(q_1, q_2, q_3)}$ ,
- (2)  $\forall p \mid (q_1, q_2), p \nmid q_3 : n \not\equiv a_1 + a_2 \pmod{p}$ ,
- (3)  $\forall p \mid (q_1, q_3), p \nmid q_2 : n \not\equiv a_1 + a_3 \pmod{p}$ ,
- (4)  $\forall p \mid (q_2, q_3), p \nmid q_1 : n \not\equiv a_2 + a_3 \pmod{p}$ .

Now condition (4) is fulfilled by the choice of  $a_2$ . We have to construct now an admissible  $a_1 \pmod{q_1}$ ,  $(a_1, q_1) = 1$ , namely such that conditions (1) – (3) are fulfilled.

Firstly,  $a_1$  has to be such that  $a_1 \equiv n - a_2 - a_3 \pmod{(q_1, q_2, q_3)}$ . Since  $n - a_2 - a_3 \equiv -h_p \not\equiv 0 \pmod{p}$  for any  $p \mid (q_2, q_3)$  we see that  $a_1 \pmod{(q_1, q_2, q_3)}$  may be chosen like that, and it will not contradict to  $(a_1, q_1) = 1$ , and also condition (1) is fulfilled.

Further  $a_1$  must be  $a_1 \equiv n - a_3 + k_p \not\equiv 0 \pmod{p}$  for every  $p \mid (q_1, q_3), p \nmid q_2$ , where  $1 \leq k_p \leq p-1$  (condition (3)), and also with  $a_1 \equiv n - a_2 + l_p \not\equiv 0 \pmod{p}$  for every  $p \mid (q_2, q_1), p \nmid q_3$ , where  $1 \leq l_p \leq p-1$  (condition (2)). Here the existence of  $l_p$  and  $k_p$  can be explained as above for  $h_p$ . Then take  $a_1$  with  $(a_1, q_1) = 1$  to hold these congruences, again via the Chinese Remainder Theorem. It is admissible by construction.  $\square$

By studying property (E), we encounter the following connection with the binary Goldbach problem.

Let  $p$  be any prime  $> 2$  and let  $n$  be sufficiently large. We can construct  $a_i, q_i$ , with  $(a_i, q_i) = 1$  for  $i = 1, 2, 3$ , and with

$$p \mid (q_1, q_2), \quad p \nmid q_3, \quad n \equiv a_1 + a_2 \pmod{p}, \quad a_1 + a_2 + a_3 \equiv n \pmod{(q_1, q_2, q_3)},$$

namely take any odd  $q_1, q_2, q_3$  such that  $p \mid (q_1, q_2)$ ,  $p \nmid q_3$ , and take  $a_1$  with  $n - a_1 \not\equiv 0 \pmod{p}$  relatively prime to  $q_1$ , take  $a_2$  relatively prime to  $q_2$  with  $a_2 \equiv n - a_1 \pmod{p}$  and  $(n - a_1 - a_2, (q_1, q_2, q_3)) = 1$ , and  $a_3$  with  $a_3 \equiv n - a_1 - a_2 \pmod{(q_1, q_2, q_3)}$  relatively prime to  $q_3$ . If we could show that there exist primes  $p_i \equiv a_i \pmod{q_i}$ ,  $i = 1, 2, 3$ , with  $n = p_1 + p_2 + p_3$ , and so  $n \equiv a_1 + a_2 + p_3 \pmod{(q_1, q_2)}$ , then since  $n \equiv a_1 + a_2 \pmod{p}$  it follows that  $0 \equiv p_3 \pmod{p}$ , so  $p_3 = p$  and  $n - p = p_1 + p_2$ . Then the number  $n - p$  would be the sum of two primes.

So if the considered ternary Goldbach problem with primes in independent arithmetic progressions touches the binary Goldbach problem, the circle method fails.

### 3 A Lemma involving sieve methods

Before considering the minor arcs we show the following Lemma by using the large sieve inequality and a formula of Montgomery in [4]. The method was already presented in [3].

**Lemma 2.** *For  $Q \geq 1$ ,  $H > 0$  and  $b_1, \dots, b_n \in \mathbb{C}$  we have*

$$\sum_{q \sim Q} q \max_{0 \leq a < q} \left| \sum_{\substack{m \leq n \\ m \equiv a \pmod{q}}} b_m \right|^2 \\ \ll (n^2 + Q^2) H^{-1} (\log Q) \max_{m \leq n} |b_m|^2 + (n + Q^2) H (\log Q) \sum_{m \leq n} |b_m|^2$$

with an absolute  $O$ -constant.

**Remark.** If  $Q$  may be some small power of  $n$  the Cauchy-Schwarz-estimate

$$\sum_{q \sim Q} q \max_{0 \leq a < q} \left| \sum_{\substack{m \leq n \\ m \equiv a \pmod{q}}} b_m \right|^2 \ll \sum_{q \leq 2Q} q \sum_{m \leq n} |b_m|^2 \frac{n}{q} \ll n Q \sum_{m \leq n} |b_m|^2$$

is weaker. An approach with the large sieve inequality involving characters does not work either.

**Proof of Lemma 2.**

For a residue class  $a \bmod q$  we set

$$N(a, q) := \sum_{\substack{m \leq n \\ m \equiv a(q)}} b_m.$$

Now the expression on the left hand side in Lemma 2 is  $E_1 + E_2$  with

$$E_1 := \sum_{\substack{q \sim Q \\ d(q) > H}} q \max_{0 \leq a < q} |N(a, q)|^2$$

and

$$E_2 := \sum_{\substack{q \sim Q \\ d(q) \leq H}} q \max_{0 \leq a < q} |N(a, q)|^2.$$

Consider first  $E_1$ . Let

$$A_Q := \#\{q; q \sim Q, d(q) > H\},$$

then

$$A_Q H < \sum_{\substack{q \sim Q \\ d(q) > H}} d(q) \leq \sum_{q \leq 2Q} d(q) \ll Q \log Q,$$

so

$$A_Q \ll \frac{Q \log Q}{H}.$$

Since  $N(a, q) \ll \left(\frac{n}{q} + 1\right) \max_{m \leq n} |b_m|$  we get

$$\begin{aligned} E_1 &\ll \sum_{\substack{q \sim Q \\ d(q) > H}} q \max_a |N(a, q)|^2 \ll \sum_{\substack{q \sim Q \\ d(q) > H}} q \left(\frac{n^2}{q^2} + 1\right) \max_{m \leq n} |b_m|^2 \\ &\ll A_Q \left(\frac{n^2}{Q} + Q\right) \max_{m \leq n} |b_m|^2 \ll \left(\frac{n^2}{H} + \frac{Q^2}{H}\right) (\log Q) \max_{m \leq n} |b_m|^2. \end{aligned}$$

This is the first summand on the right hand side of Lemma 2.

Now to  $E_2$ .

For any integer  $0 \leq h < q$  let

$$f_h(q) := \sum_{d|q} \mu(d) \frac{q}{d} N\left(h, \frac{q}{d}\right),$$

so Möbius' inversion formula gives

$$qN(h, q) = \sum_{d|q} f_h(d)$$

for all  $0 \leq h < q$ . With this we have

$$\begin{aligned} E_2 &= \sum_{\substack{q \sim Q \\ d(q) \leq H}} \frac{1}{q} \max_{0 \leq a < q} q^2 |N(a, q)|^2 = \sum_{\substack{q \sim Q \\ d(q) \leq H}} \frac{1}{q} \max_{0 \leq a < q} \left| \sum_{d|q} f_a(d) \right|^2 \\ &\leq \sum_{\substack{q \sim Q \\ d(q) \leq H}} \frac{d(q)}{q} \sum_{d|q} \max_{0 \leq a < q} |f_a(d)|^2. \end{aligned}$$

The maximum is taken over  $a$  with  $0 \leq a < q$ . We see that  $|f_a(d)|^2$  is  $d$ -periodic in  $a$  for  $d|q$ , since  $N(a+d, t) = N(a, t)$  for  $t|d$ , so

$$f_{a+dl}(d) = \sum_{t|d} \mu(t) \frac{d}{t} N\left(a+dl, \frac{d}{t}\right) = \sum_{t|d} \mu(t) \frac{d}{t} N\left(a, \frac{d}{t}\right) = f_a(d) \text{ for all } l \in \mathbb{Z},$$

therefore the maximum stays equal if taken only over  $a$  with  $0 \leq a < d$ . We estimate this maximum by  $\sum_{0 \leq a < d}$  and get

$$E_2 \leq \sum_{\substack{q \sim Q \\ d(q) \leq H}} \frac{d(q)}{q} \sum_{d|q} \sum_{0 \leq a < d} |f_a(d)|^2.$$

By Montgomery in [4], equation (10), we have for  $T(\alpha) := \sum_{m \leq n} b_m e(\alpha m)$ ,  $\alpha \in \mathbb{R}$ , the formula

$$\frac{1}{d} \sum_{h=0}^{d-1} |f_h(d)|^2 = \sum_{\substack{a \leq d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right) \right|^2,$$



that we can apply here. We get

$$\begin{aligned}
E_2 &\leq \sum_{\substack{q \sim Q \\ d(q) \leq H}} d(q) \sum_{d|q} \frac{d}{q} \sum_{\substack{a < d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right) \right|^2 \\
&\leq H \sum_{d \leq 2Q} \left( \sum_{\substack{q \sim Q \\ d|q}} \frac{d}{q} \right) \sum_{\substack{a < d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right) \right|^2 \\
&\ll H(\log Q) \sum_{d \leq 2Q} \sum_{\substack{a < d \\ (a,d)=1}} \left| T\left(\frac{a}{d}\right) \right|^2 \\
&\ll H(\log Q) (n + Q^2) \sum_{m \leq n} |b_m|^2
\end{aligned}$$

by the inequality of the large sieve. This is the second term on the right hand side of Lemma 2.  $\square$

## 4 The conclusion with Lemma 2

Now let  $A, \theta > 0$  and  $\vartheta > 0$  as above. Let  $Q_1, Q_2, Q_3 \leq n^{1/2}/(\log n)^\vartheta$ .

We consider first

$$\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} := \sum_{q_3 \sim Q_3} \max_{a_3} \sum_{q_2 \sim Q_2} \max_{a_2} \sum_{q_1 \sim Q_1} \max_{a_1} |J_3^{\mathfrak{m}}(n)|.$$

From the definition of  $J_3$  and  $J_2$  we have

$$\begin{aligned}
\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} &\leq \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \sum_{\substack{m_1 \leq n \\ m_1 \equiv a_1(q_1)}} \Lambda(m_1) |J_2^{\mathfrak{m}}(n - m_1)| \\
&\leq \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \sum_{\substack{m \leq n \\ m \equiv n - a_1(q_1)}} (\log n) |J_2^{\mathfrak{m}}(m)|.
\end{aligned}$$

By Cauchy-Schwarz' inequality we now get

$$\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} \leq (\log n) \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \left( \sum_{q_1 \sim Q_1} q_1 \max_{a_1} \left| \sum_{\substack{m \leq n \\ m \equiv a_1(q_1)}} |J_2^{\mathfrak{m}}(m)| \right|^2 \right)^{1/2}$$

and we apply Lemma 2 to the expression in large brackets.

Since  $Q_1 \leq n^{1/2}$  we see that

$$\begin{aligned} \mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} &\ll (\log n) \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \left( \frac{n^2}{H} (\log n) \max_{m \leq n} |J_2^{\mathfrak{m}}(m)|^2 \right. \\ &\quad \left. + nH(\log n) \sum_{m \leq n} |J_2^{\mathfrak{m}}(m)|^2 \right)^{1/2}. \end{aligned}$$

Now we apply the following two lemmas, which will be proven in the last paragraphs.

**Lemma 3.** *For  $Q_2, Q_3 \leq n^{1/2}/(\log n)^\vartheta$  we have*

$$\sum_{q_2, q_3} \max_{a_2, a_3} |J_2^{\mathfrak{m}}(m)| \ll n(\log n)^7.$$

**Lemma 4.** *For  $Q_2 \leq n^{1/2}/(\log n)^\vartheta$  and  $Q_3 \leq (\log n)^\theta$  we have*

$$\sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \left( \sum_{m \leq n} |J_2^{\mathfrak{m}}(m)|^2 \right)^{\frac{1}{2}} \ll \frac{n^{3/2}}{(\log n)^{2A+16}}.$$

Here the sum over such a small  $Q_3$ -range is of course pointless; but we state it here to see why no larger bound for  $Q_3$  is possible to get with the given method in the proof of Lemma 4.

With  $H := (\log n)^{2A+23}$  it follows from Lemma 3 and 4 that

$$\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} \ll \frac{n^2}{(\log n)^{A+3}}.$$

Finally, together with Theorem 3, we get for  $Q_2 \leq n^{1/2}/(\log n)^\vartheta$  and  $Q_3 \leq (\log n)^\theta$  the estimate

$$\begin{aligned} &\sum_{q_3 \sim Q_3} \max_{\substack{a_3 \\ (a_3, q_3)=1}} \sum_{q_2 \sim Q_2} \max_{\substack{a_2 \\ (a_2, q_2)=1}} \sum_{q_1 \sim Q_1} \max_{\substack{a_1 \\ (a_1, q_1)=1}} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \\ &\leq \sum_{\substack{k, Q_3=2^k \\ \leq (\log n)^\theta}} \sum_{\substack{j, Q_2=2^j \\ \leq n^{1/2}/(\log n)^\vartheta}} \sum_{\substack{i, Q_1=2^i \\ \leq n^{1/2}/(\log n)^\vartheta}} (\mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}} + \mathcal{E}_{Q_1, Q_2, Q_3}^{\mathfrak{m}}) \end{aligned}$$

$$\ll (\log n)^3 \cdot \frac{n^2}{(\log n)^{A+3}} = \frac{n^2}{(\log n)^A},$$

and from that follows Theorem 1.

So it remains to show Lemma 3 and Lemma 4.

## 5 Two Lemmas on the minor arcs

### 5.1 Proof of Lemma 3

We have

$$\sum_{q_2, q_3} \max_{\substack{m \leq n \\ a_2, a_3}} |J_2^{\mathfrak{m}}(m)| = \sum_{q_2, q_3} \max_{\substack{m \leq n \\ a_2, a_3}} |J_2(m) - J_2^{\mathfrak{M}}(m)|.$$

Now we estimate  $J_2(m)$  and  $J_2^{\mathfrak{M}}(m)$ . The reason why we split  $J_2^{\mathfrak{m}}(m)$  is that the trivial upper estimate for  $J_2^{\mathfrak{m}}(m)$ , namely

$$J_2^{\mathfrak{m}}(m) \ll \int_0^1 |S_2(\alpha) S_3(\alpha)| d\alpha,$$

does not suffice.

We have

$$\begin{aligned} J_2(m) &= \int_0^1 S_2(\alpha) S_3(\alpha) e(-m\alpha) d\alpha \\ &= \sum_{\substack{m_2 \leq n \\ m_2 \equiv a_2 \pmod{q_2}}} \Lambda(m_2) \sum_{\substack{m_3 \leq n \\ m_3 \equiv a_3 \pmod{q_3}}} \Lambda(m_3) \int_0^1 e(\alpha(m_2 + m_3 - m)) d\alpha \end{aligned}$$

and by the orthogonal relations for  $e(\alpha m)$  we have that the last integral is 1, if  $m_2 + m_3 = m$ , and 0 otherwise. Therefore we get

$$J_2(m) \ll \sum_{\substack{m_2 \leq n \\ m_2 \equiv a_2 \pmod{q_2} \\ m_2 \equiv m - a_3 \pmod{q_3}}} (\log n)^2 \ll \frac{n}{[q_2, q_3]} (\log n)^2 \ll \frac{n}{q_2 q_3} (\log n)^2 (q_2, q_3),$$

so

$$\sum_{q_2, q_3} \max_{\substack{m \leq n \\ a_2, a_3}} |J_2(m)| \ll n (\log n)^2 \sum_{q_2, q_3} \frac{(q_2, q_3)}{q_2 q_3}$$

$$\ll n(\log n)^2 \sum_{m, q'_2, q'_3} \frac{m}{mq'_2mq'_3} \ll n(\log n)^5.$$

Now we consider the following

**Proposition 4.** *We have*

$$\sum_{q_2, q_3} \max_{\substack{a_2, a_3 \\ m \leq n}} |J_2^{\mathfrak{M}}(m)| \ll n(\log n)^7.$$

By this and together with above estimation we get therefore Lemma 3.  $\square$

**Proof of Proposition 4.**

We have to consider the analogous estimation for  $J_2^{\mathfrak{M}}(m)$  as was done in paragraph 2.1 in order to estimate  $J_3^{\mathfrak{M}}(m)$ .

We get

$$J_2^{\mathfrak{M}}(m) = \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} I(a, q)$$

with

$$\begin{aligned} I(a, q) &= \int_{-R/qn}^{R/qn} S_2\left(\frac{a}{q} + \alpha\right) S_3\left(\frac{a}{q} + \alpha\right) e\left(-m\left(\frac{a}{q} + \alpha\right)\right) d\alpha \\ &= \frac{(c_2 c_3)(a, q)}{\varphi([q_2, q])\varphi([q_3, q])} e\left(-m\frac{a}{q}\right) \int_{-R/qn}^{R/qn} M^2(\alpha) e(-m\alpha) d\alpha \\ &\quad + \sum_{i, j} \frac{1}{\varphi([q_i, q])} \int_{-R/qn}^{R/qn} |M(\alpha)| d\alpha \cdot O\left(\frac{R}{q} (\log n)^2 \Delta(n, [q_j, q])\right) \\ &\quad + O\left(\frac{R^3}{nq^3} (\log n)^4 \Delta(n, [q_2, q]) \Delta(n, [q_3, q])\right) \\ &=: H_{a, q}(m) + \mathcal{O}_1 + \mathcal{O}_2, \end{aligned}$$

say. Now

$$\begin{aligned} \sum_{q_2, q_3} \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \mathcal{O}_1 &\ll \sum_{i, j} \sum_{q \leq R} \sum_{q_i} \frac{1}{\varphi([q_i, q])} \sum_{\substack{a < q \\ (a, q) = 1}} \frac{R}{q} (\log n)^2 \sum_{q_j} \Delta(n, [q_j, q]) \\ &\ll \sum_{i, j} \sum_{q_i} \frac{\log \log n}{q_i} R (\log n)^2 \sum_{q_j} \sum_{q \leq R} \Delta(n, [q_j, q]) \end{aligned}$$

$$\ll (\log n)^4 R \sum_j \sum_{h_j \leq RQ_j} \omega(h_j) \Delta(n, h_j)$$

with

$$\begin{aligned} \omega(h_j) &:= \sum_{q_j} \sum_{\substack{q \leq R \\ [q_j, q] = h_j}} 1 = \sum_{d_j \leq R} \sum_{q_j} \sum_{\substack{q \leq R \\ (q, q_j) = d_j \\ qq_j = h_j d_j}} 1 \\ &\ll \sum_{d_j \leq R} \sum_{\substack{q \leq R \\ d_j | q}} 1 \ll R \log R \ll R \log n. \end{aligned}$$

So the  $\mathcal{O}_1$ -error term is

$$\begin{aligned} &\ll R^2 (\log n)^5 \sum_j \sum_{h_j \leq RQ_j} \Delta(n, h_j) \ll R^2 (\log n)^5 \cdot \frac{n}{(\log n)^{\vartheta - B - 6}} \\ &\ll n (\log n)^{3B - \vartheta + 11} \ll n (\log n)^{-A - B - 2} \ll n, \end{aligned}$$

again by using Bombieri-Vinogradov's Theorem and  $\vartheta \geq A + 4B + 13$ .

Now to  $\mathcal{O}_2$ . We have

$$\begin{aligned} \sum_{q_2, q_3} \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \mathcal{O}_2 &\ll \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \frac{R^3}{nq^3} (\log n)^4 \sum_{q_2, q_3} \Delta(n, [q_2, q]) \Delta(n, [q_3, q]) \\ &\ll \frac{R^3}{n} (\log n)^4 \sum_{\substack{h_2 \leq RQ_2 \\ h_3 \leq RQ_3}} \omega(h_2, h_3) \Delta(n, h_2) \Delta(n, h_3) \end{aligned}$$

with

$$\begin{aligned} \omega(h_2, h_3) &:= \sum_{q_2, q_3} \sum_{\substack{q \leq R \\ [q_i, q] = h_i \\ i=2,3}} \frac{1}{q^2} = \sum_{d_2, d_3 \leq R} \sum_{q_2, q_3} \sum_{\substack{q \leq R \\ (q_i, q) = d_i \\ q_i q = h_i d_i \\ i=2,3}} \frac{1}{q^2} \\ &\ll \sum_{d_2, d_3 \leq R} \sum_{\substack{q \leq R \\ [d_2, d_3] | q}} \frac{1}{q^2} \ll \sum_{d_2, d_3} \sum_{q \leq R} \frac{1}{q^2 [d_2, d_3]^2} \\ &= \sum_{d_2, d_3} \sum_{q \leq R} \frac{1}{q^2 d_2^2 d_3^2} (d_2, d_3)^2 \ll \sum_{d_3 \leq R} 1 \ll R, \end{aligned}$$

so the  $\mathcal{O}_2$ -error term is

$$\ll \frac{R^4}{n} (\log n)^4 \left( \sum_{h_2 \leq RQ_2} \Delta(n, h_2) \right) \left( \sum_{h_3 \leq RQ_3} \Delta(n, h_3) \right) \ll n (\log n)^{6B-2\vartheta+16} \ll n,$$

again by using Bombieri-Vinogradov's Theorem and  $\vartheta \geq A + 4B + 13$ .

Now there remains the main term. Since

$$\int_{-R/qn}^{R/qn} M^2(\alpha) e(-m\alpha) d\alpha = m - 1 + O\left(\frac{qn}{R}\right) \ll n$$

for  $q \leq R$  we can estimate it in the following way. It is

$$\begin{aligned} H &:= \sum_{q_2, q_3} \max_{\substack{m \leq n \\ a_2, a_3}} \sum_{q \leq R} \sum_{\substack{a < q \\ (a, q) = 1}} \frac{(c_2 c_3)(a, q)}{\varphi([q_2, q]) \varphi([q_3, q])} e\left(-m \frac{a}{q}\right) \int_{-R/qn}^{R/qn} M^2(\alpha) e(-m\alpha) d\alpha \\ &\ll n \sum_{q_2, q_3} \sum_{q \leq R} \frac{q(\log n)}{[q_2, q][q_3, q]} = n(\log n) \sum_{q_2, q_3} \sum_{q \leq R} \frac{(q_2, q)(q_3, q)}{q_2 q q_3} \\ &\ll n(\log n) \sum_{a, b, c, d, e, f, g} \frac{dc \cdot db}{dace \cdot dabf \cdot dbcg} \ll n(\log n)^7, \end{aligned}$$

where we substituted  $q_2 = dace$ ,  $q_3 = dabf$ ,  $q = dbcg$  with  $a, b, c, d, e, f, g \leq n$ ,  $d := (q, q_1, q_3)$ , and pairwise relatively prime  $a, b, c$  and  $e, f, g$ .

This shows the Proposition.  $\square$

## 5.2 Proof of Lemma 4

Since the left hand side of Lemma 4 is

$$\ll \left( \sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \sum_{m \leq n} |J_2^m(m)|^2 \right)^{\frac{1}{2}},$$

it suffices to show that

$$\sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \sum_{m \leq n} |J_2^m(m)|^2 \ll \frac{n^3}{(\log n)^{4A+32}}$$

for any  $A > 0$  in the required regions for  $Q_2$  and  $Q_3$ . The left hand side is

$$\begin{aligned} & \sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \sum_{m \leq n} \left| \int_{\mathfrak{m}} S_2(\alpha) S_3(\alpha) e(-m\alpha) d\alpha \right|^2 \\ & \leq \sum_{q_3} q_3 \max_{a_3} \sum_{q_2} q_2 \max_{a_2} \int_{\mathfrak{m}} |S_2(\alpha) S_3(\alpha)|^2 d\alpha \end{aligned}$$

by Bessel's inequality. Now

$$\begin{aligned} |S_2(\alpha)|^2 &= \sum_{\substack{m, m' \leq n \\ m \equiv m' \equiv a_2 \pmod{q_2}}} \Lambda(m) \Lambda(m') e(\alpha(m - m')) \\ &= \sum_{\substack{|r| \leq n \\ r \equiv 0 \pmod{q_2}}} e(\alpha r) \sum_{\substack{m \leq n \\ m \equiv a_2 \pmod{q_2} \\ m - r \leq n}} \Lambda(m) \Lambda(m - r) \\ &=: \sum_{\substack{|r| \leq n \\ r \equiv 0 \pmod{q_2}}} e(\alpha r) R(r; a_2, q_2), \end{aligned}$$

say, with  $R(r; a_2, q_2) \ll \frac{n}{q_2} (\log n)^2$ .

So the left hand side is

$$\begin{aligned} & \ll n(\log n)^2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \sum_{q_2 \sim Q_2} \sum_{\substack{|r| \leq n \\ r \equiv 0 \pmod{q_2}}} \left| \int_{\mathfrak{m}} |S_3(\alpha)|^2 e(\alpha r) d\alpha \right| \\ & \ll n(\log n)^2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \sum_{0 < |r| \leq n} \tau(|r|) \left| \int_{\mathfrak{m}} |S_3(\alpha)|^2 e(\alpha r) d\alpha \right| \\ & + n(\log n)^2 Q_2 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \int_0^1 |S_3(\alpha)|^2 d\alpha. \end{aligned}$$

Now

$$\int_0^1 |S_3(\alpha)|^2 d\alpha \ll \frac{n}{q_3} (\log n)^2,$$

so the second term is  $\ll n^2 (\log n)^2 Q_2 Q_3 (\log n)^2 \ll n^{5/2} (\log n)^{4+\theta} \ll n^3 (\log n)^{-A}$  and therefore in the required bound.

The first term is

$$\begin{aligned}
&\ll n(\log n)^2 \sum_{q_3} q_3 \max_{a_3} \left( \sum_{0 < |r| \leq n} \tau(|r|)^2 \right)^{1/2} \left( \sum_{0 < |r| \leq n} \left| \int_{\mathfrak{m}} |S_3(\alpha)|^2 e(\alpha r) d\alpha \right|^2 \right)^{1/2} \\
&\ll n^{3/2} (\log n)^4 \sum_{q_3 \sim Q_3} q_3 \max_{a_3} \left( \int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \right)^{1/2} \\
&\ll n^{3/2} (\log n)^4 \left( \sum_{q_3 \sim Q_3} q_3^2 \right)^{1/2} \left( \sum_{q_3 \sim Q_3} \max_{a_3} \int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \right)^{1/2} \\
&\ll n^{3/2} (\log n)^4 \left( \sum_{q_3 \sim Q_3} q_3^3 \max_{a_3} \int_{\mathfrak{m}} |S_3(\alpha)|^4 d\alpha \right)^{1/2}.
\end{aligned}$$

Now here is the difficulty to show a nontrivial bound for the expression in large brackets. It should be  $\ll n^3/(\log n)^C$  for any large constant  $C > 0$  and large  $Q_3$ , but however one tries to manage it, there is still some power of  $Q_3$  left. We best can give the bound

$$\ll n^{3/2} (\log n)^4 \left( \sum_{q_3 \sim Q_3} q_3^3 \max_{a_3} \max_{\alpha \in \mathfrak{m}} |S_3(\alpha)|^2 \int_0^1 |S_3(\alpha)|^2 d\alpha \right)^{1/2}.$$

Now we need another Lemma to estimate  $|S_3(\alpha)|^2$  for  $\alpha \in \mathfrak{m}$ , it is the following.

**Lemma 5.** *For all  $q_3 \sim Q_3$ ,  $(a_3, q_3) = 1$  and  $\alpha \in \mathfrak{m}$  we have  $|S_3(\alpha)|^2 \ll \frac{n^2}{q_3(\log n)^C}$  for  $C = 8A + 2\theta + 74$ .*

By using this we get for the above expression

$$\begin{aligned}
&\ll n^{3/2} (\log n)^5 \left( \sum_{q_3 \sim Q_3} Q_3 \frac{n^3}{(\log n)^C} \right)^{1/2} \\
&\ll \frac{n^3}{(\log n)^{C/2-5}} Q_3 \ll \frac{n^3}{(\log n)^{4A+32}}
\end{aligned}$$

for  $C = 8A + 2\theta + 74$  since  $Q_3 \leq (\log n)^\theta$ .

So we see that  $Q_3$  cannot be chosen as a power of  $n$  using the given method.



But this estimation shows Lemma 4 for  $Q_2 \leq n^{1/2}/(\log n)^\vartheta$  and  $Q_3 \leq (\log n)^\theta$  as required.  $\square$

**Proof of Lemma 5.**

By Lemma 2 of A. Balog in [1] we have the validity of the following assertion. For  $C > 0$  there exists a  $D = D(C) > 0$  such that for any  $\alpha \in \mathbb{R}$  with  $\|\alpha - \frac{u}{v}\| < \frac{1}{v^2}$  with integers  $(u, v) = 1$  and  $(\log n)^D \leq v \leq \frac{n}{(\log n)^D}$  we have

$$\sum_{q_3 \leq n^{1/3}/(\log n)^D} q_3 \max_{(a_3, q_3)=1} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^C},$$

and since  $Q_3 \leq (\log n)^\theta \ll \frac{n^{1/3}}{(\log n)^D}$  also

$$\sum_{q_3 \sim Q_3} q_3 \max_{(a_3, q_3)=1} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^C}.$$

By Dirichlet's Approximation Theorem, for  $\alpha \in \mathbb{R}$  and  $B > 0$  there exist integers  $u, v$ ,  $1 \leq v \leq n/(\log n)^B$ , with  $(u, v) = 1$  and  $\|\alpha - \frac{u}{v}\| < \frac{(\log n)^B}{vn}$ , and for  $\alpha \in \mathfrak{m}$  it follows that  $v \geq (\log n)^B$ .

Therefore the conditions of Balog's Lemma are fulfilled if we take  $B \geq D(8A + 2\theta + 74)$ , and it can be applied then. It follows that for all  $\alpha \in \mathfrak{m}$  we have

$$\sum_{q_3 \sim Q_3} q_3 \max_{a_3} |S_3(\alpha)|^2 \ll \frac{n^2}{(\log n)^{8A+2\theta+74}},$$

and so we have for all  $q_3 \sim Q_3$  and  $(a_3, q_3) = 1$  the inequality

$$|S_3(\alpha)|^2 \ll \frac{n^2}{q_3 (\log n)^{8A+2\theta+74}},$$

since

$$|S_3(\alpha)|^2 \ll \frac{1}{Q_3} \sum_{q_3 \sim Q_3} q_3 \max_{a_3} |S_3(\alpha)|^2 \ll \frac{1}{Q_3} \cdot \frac{n^2}{(\log n)^{8A+2\theta+74}}.$$

That shows Lemma 5.  $\square$

## 6 Proof of Theorem 2

Now we prove Theorem 2 in this last section. Let  $A, \theta, \vartheta > 0$  be as in Theorem 2 and let  $n$  be odd and sufficiently large.

Besides  $J_3(n)$  consider also

$$R_3(n) = \sum_{\substack{p_1, p_2, p_3 \\ p_1 + p_2 + p_3 = n \\ p_i \equiv a_i (q_i), \\ i=1,2,3}} \log p_1 \log p_2 \log p_3 \quad \text{and} \quad r_3(n) = \sum_{\substack{p_1, p_2, p_3 \\ p_1 + p_2 + p_3 = n \\ p_i \equiv a_i (q_i), \\ i=1,2,3}} 1.$$

Then we have

$$|R_3(n) - J_3(n)| \leq (\log n)^3 W,$$

where  $W$  denotes the number of solutions of  $p^l + q^j + r^k = n$ , with  $p, q, r$  prime and where  $l, j$  or  $k$  are at least 2, and  $p^l \equiv a_1 (q_1)$ ,  $q^j \equiv a_2 (q_2)$ ,  $r^k \equiv a_3 (q_3)$ . Now four cases occur: For  $i = 1, 2, 3, 4$  let  $W_{(i)}$  be the number of solutions in case (i), namely (1)  $l, j \geq 2$ , (2)  $l = 1, j \geq 2$ , (3)  $l \geq 2, j = 1$ , (4)  $l = j = 1, k \geq 2$ .

In case (1) there are at most  $O(\sqrt{n})$  many possibilities for  $p^l, q^j \leq n$ , so  $W_{(1)} \ll n$  and we have  $\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W_{(1)} \ll \frac{n^2}{(\log n)^{2\vartheta - \theta}} \ll \frac{n^2}{(\log n)^{A+3}}$  since  $\vartheta > \theta + A + 3$ .

In case (4) we have at most  $O(\sqrt{n})$  many possibilities for  $r^k \leq n$  and  $\ll \frac{n}{q_2}$  many for  $q$ , so  $W_{(4)} \ll \frac{n^{3/2}}{q_2}$  and we get  $\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W_{(4)} \ll Q_1 Q_3 n^{3/2} \ll \frac{n^2}{(\log n)^{\vartheta - \theta}} \ll \frac{n^2}{(\log n)^{A+3}}$  since  $\vartheta > \theta + A + 3$ .

The same estimation comes of course analogously with  $W_{(2)}$  in case (2).

In case (3) we consider the number

$$\#\{p^l \leq n; l \geq 2, p^l \equiv a_1 (q_1)\} \leq \sum_{\substack{m \leq n \\ m \equiv a_1 (q_1)}} \Lambda(m)(1 - \mu^2(m)) =: N(a_1, q_1)$$

in the context of section 3, with  $b_m := \Lambda(m)(1 - \mu^2(m))$ . Then  $W_{(3)} \ll N(a_1, q_1) \cdot \frac{n}{q_2}$ , and by application of Lemma 2 we get

$$\begin{aligned} \sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W_{(3)} &\ll n Q_3 \sum_{q_1} \max_{a_1} N(a_1, q_1) \ll n Q_3 \left( \sum_{q_1} q_1 \max_{a_1} N(a_1, q_1)^2 \right)^{1/2} \\ &\ll n Q_3 \left( \frac{n^2}{H} + n^{3/2} H \right)^{1/2} \log n \end{aligned}$$

since

$$\sum_{m \leq n} |b_m|^2 = \sum_{\substack{m \leq n \\ m \equiv a_1 \pmod{q_1}}} \Lambda^2(m)(1 - \mu^2(m))^2 \ll \sum_{\substack{p^k \leq n \\ k \geq 2}} (\log p)^2 \ll \sqrt{n} \log n$$

and  $Q_1 \leq \sqrt{n}$ .

If we choose the parameter  $H$  as  $H := \frac{n^{1/2}}{(\log n)^{2A+6} Q_3^2}$  we get further

$$\begin{aligned} \sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W_{(3)} &\ll n Q_3 \left( n^{3/2} (\log n)^{2A+6} Q_3^2 + \frac{n^2}{Q_3^2 (\log n)^{2A+6}} \right)^{1/2} \\ &\ll n \cdot n^{3/4} Q_3^2 (\log n)^{A+3} + \frac{n^2}{(\log n)^{A+3}} \ll \frac{n^2}{(\log n)^{A+3}}. \end{aligned}$$

So we get

$$\sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W \ll \frac{n^2}{(\log n)^{A+3}}.$$

Therefore it follows from Theorem 1:

$$\begin{aligned} &\sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \left| R_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \\ &\leq \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \left| R_3(n) - J_3(n) \right| \\ &\quad + \sum_{q_3} \max_{a_3} \sum_{q_2} \max_{a_2} \sum_{q_1} \max_{a_1} \left| J_3(n) - \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \\ &\ll \sum_{q_1, q_2, q_3} \max_{a_1, a_2, a_3} W (\log n)^3 + \frac{n^2}{(\log n)^A} \ll \frac{n^2}{(\log n)^A}. \end{aligned}$$

So the formula of Theorem 1 holds also for  $R_3(n)$  instead of  $J_3(n)$ .

Now let  $q_3 \leq Q_3 = (\log n)^\theta$  and  $(a_3, q_3) = 1$  be fixed.

For given  $q_2$  and admissible  $a_2$  consider

$$\mathcal{Q}_1 := \{q_1 \leq Q_1; \exists a_1 \text{ adm. : } R_3(n) = 0\}, \quad E_1 := \#\mathcal{Q}_1,$$

and

$$\mathcal{Q}_2 := \{q_2 \leq Q_2; \exists a_2 \text{ adm. : } E_1 \geq Q_1 (\log n)^{-A}\}, \quad E_2 := \#\mathcal{Q}_2.$$

We have  $\mathcal{S}_3(n) \gg 1$  if it is positive (see the formula for it as Euler product), so we have

$$\begin{aligned}
E_2 \cdot \frac{Q_1}{(\log n)^A} \cdot \frac{n^2}{Q_1 Q_2 Q_3} \\
\leq \sum_{q_2 \in \mathcal{Q}_2} \max_{\substack{a_2 \text{ adm.} \\ E_1 \geq \frac{Q_1}{(\log n)^A}}} \sum_{q_1 \in \mathcal{Q}_1} \max_{\substack{a_1 \text{ adm.} \\ R_3(n)=0}} \left| \frac{n^2 \mathcal{S}_3(n)}{2\varphi(q_1)\varphi(q_2)\varphi(q_3)} \right| \\
\ll \frac{n^2}{(\log n)^{2A+\theta}}
\end{aligned}$$

by Theorem 1, and it follows that  $E_2 \ll Q_2(\log n)^{-A}$ .

So for almost all  $q_2$  and all admissible  $a_2$  we have that  $E_1 < Q_1(\log n)^{-A}$ , that means that for almost all  $q_1$  and all admissible  $a_1$  it holds that  $R_3(n) > 0$ . Since  $r_3(n) \geq \frac{R_3(n)}{(\log n)^3}$ , it follows that  $r_3(n)$  is positive, too, so Theorem 2 follows.  $\square$

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